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Research Statement

My research is in discrete mathematics; specifically, spectral hypergraph theory, computational algebra, crossing numbers, and network modeling. A central theme of my dissertation work is exploring the characteristic polynomial of a hypergraph which is connected to numerous fields of mathematics because this polynomial is the resultant of a system of multilinear homogeneous equations. The resultant, which is a generalization of the determinant, is central to algebraic geometry, commutative algebra, and numerical multilinear algebra among other areas. My dissertation gives a generalization of the Harary-Sachs Theorem by providing a combinatorial description of the coefficients of characteristic polynomial of a hypergraph. This result is valuable because we can compute the leading coefficients of this particular polynomial without needing to compute all the coefficients, which is NP-hard to compute in general. Furthermore, I have provided a numerically stable algorithm which can compute the characteristic polynomial of a hypergraph given its set spectrum and leading coefficients. This allows for the computation of the characteristic polynomial of a hypergraph when traditional tools from commutative algebra (i.e., the resultant) have been insufficient. In addition to my dissertation work I have also collaborated on the study of crossing number problems and have applied my research in hypergraphs to the study of online black markets.

1 Spectral Hypergraph Theory

Given a graph G we would like to understand its structure. One can define a certain matrix of a graph (e.g., adjacency, incidence, Laplacian, etc.) and connect the multiset of eigenvalues of this matrix back to the graph. For example, the existence of strongly regular graphs (i.e., a regular graph where all pairs of adjacent vertices have a common neighbors and all pairs of non-adjacent vertices have b common neighbors) is sharply constrained by the graph's spectrum. Famously, the Hoffman-Singleton theorem, whose proof integrally uses a delicate analysis of a strongly regular graph's spectral properties, says that the only graphs with girth 5 and diameter 2 are necessarily d-regular for $d \in \{2, 3, 7, 57\}$. A construction for the $d = 57$ case remains open. Spectral graph theory has also found applications to real-world problems, for example, in the identification of key users in a social network via the eigencentrality measure. This idea is central to Google's PageRank patent. With the recent surge of interest in data science, network analysts are considering more complex sets of data. In many cases one may want to capture salient properties of a network where connections can be better captured through group connections. To do so, we extend these results to *hypergraphs*.

My dissertation research is in spectral hypergraph theory, where we relate the structure of a hypergraph to its spectrum and vice versa. A hypergraph is a generalization of a graph wherein edges potentially contain more than two vertices. Hypergraphs allow one to model complex systems with greater fidelity: committees of representatives, similarities between friends, and the evolution of a network over time. The cost of analyzing this richer structure is paid for in theoretical and computational complexity. Where the adjacency characteristic polynomial of a graph can be quickly computed as the determinant of a matrix, the adjacency characteristic polynomial of a hypergraph is the resultant of a multilinear homogeneous system of equations [7] (which is known to be NP-hard to compute in general [11]). My research in spectral hypergraph theory has focused on understanding the adjacency characteristic polynomial of a hypergraph. To this end we have generalized the Harary-Sachs Theorem to hypergraphs, characterized the spectrum of hypertrees, and provided a numerically stable algorithm for computing the characteristic polynomial of a hypergraph given its set of eigenvalues and some leading coefficients. Below I discuss the motivation, impact, and my future plans in spectral hypergraph theory.

1.1 A Generalization of the Harary-Sachs Theorem to Hypergraphs

An early, seminal result in spectral graph theory of Harary [12] (and later, more explicitly, Sachs [18]) expressed the coefficients of a graph's characteristic polynomial as a certain weighted sum of the counts of various subgraphs of G.

Theorem 1. *(Harary-Sachs Theorem) Let G be a labeled simple graph on n vertices. If H_i denotes the collection of i-vertex graphs whose components are edges or cycles, and c_i denotes the codegree-i coefficient of the characteristic polynomial of G (i.e., the coefficient of λ^{n-i}), then*

$$c_i = \sum_{H \in H_i} (-1)^{c(H)} 2^{z(H)} [\# H \subseteq G]$$

where $c(H)$ is the number of components of H, $z(H)$ is the number of components which are cycles, and $[\# H \subseteq G]$ denotes the number of (labeled) subgraphs of G which are isomorphic to H.

The Harary-Sachs Theorem relates the spectrum of a graph and its elementary subgraphs (i.e., disjoint union of edges and cycles). We have generalized this theorem to hypergraphs and provided an analogous description of elementary subgraphs which we refer to as *Veblen graphs.* This result allows us to compute partial information about the characteristic polynomial of a hypergraph when computing the whole polynomial is computationally costly. We are excited to have generalized the Harary-Sachs Theorem to hypergraphs in the following way.

Theorem 2. *Let $\mathcal{H}$ be a k-uniform hypergraph on n vertices. If $\mathcal{V}_i$ denotes the set of k-uniform Veblen multi-hypergraphs (i.e., all vertices have degree divisible by k), and c_i denotes the codegree-i coefficient in the characteristic polynomial of $\mathcal{H}$, then*

$$c_i = \sum_{H \in \mathcal{V}_i(\mathcal{H})} (-(k-1)^n)^{c(H)} C_H (\# H \subseteq \mathcal{H})$$

where $c(H)$ is the number of components of H, C_H is a certain computable coefficient of H, and $(\# H \subseteq \mathcal{H})$ is the number of particular maps of H to subgraphs of $\mathcal{H}$.

Theorem 2 is an authentic generalization of the Harary-Sachs Theorem as Theorem 2 simplifies to the Harary-Sachs Theorem when $\mathcal{H}$ is a graph (i.e., a 2-uniform hypergraph). We point out that this result implies that the spectrum of a hypergraph is computable from the counts of its Veblen subgraphs, just as the counts of its elementary subgraphs determine the spectrum of a graph. While this situation for hypergraphs is predictably more complicated than the graph case, it does directly connect the characteristic polynomial of a hypergraph to its structure.

1.2 The Spectrum of Hypertrees

When studying the spectrum of a graph, a simple family to consider is the collection of trees. As an example, one can apply the Harary-Sachs Theorem to show that the coefficients of the characteristic polynomial of a tree count the number of matchings of a particular size. This yields that multiplicity of the zero eigenvalue of a tree is equal to the size of its largest matching. Surprisingly, a similar bound on the multiplicity of the zero eigenvalue for an arbitrary graph continues to elude description. We discuss our results concerning the spectrum of a hypertree which are stronger than their tree analogue. We demonstrate the peculiarity of this dichotomy by providing a spectral characterization of so-called "power trees".

In [4] we show that the spectrum of a hypertree is the collection of all totally non-zero eigenvalues of its subtrees. An eigenvalue is *totally non-zero* if the eigenvalue is non-zero and corresponds to an eigenvector with all non-zero entries. In [20] the authors show that the totally non-zero eigenvalues of a hypertree are roots of a certain matching polynomial. **Using their formula we characterize the set spectrum of a hypertree as the union of the totally non-zero eigenvalues of its induced subtrees**. In other words, we show that the totally non-zero eigenvalues of a hypertree are necessarily eigenvalues of any hypertree which contains it as a subgraph. Note that this is a variant of the Cauchy Interlacing Theorem which says that the eigenvalues of a subgraph (formed by removing one vertex) interlace the eigenvalues of the original graph. This is somewhat surprising in light of the fact that the same statement is not true for ordinary graphs. We demonstrate this peculiarity by considering power trees. A *power tree* is a k-uniform hypergraph created by adding $k-2$ new vertices to each edge of a tree. We have shown that a power tree is characterized by its eigenvalues being cyclotomic.

1.3 Stably Computing the Multiplicity of Known Roots

The characteristic polynomial of a k-uniform hypergraph with n vertices is the univariate (in λ) polynomial obtained from the resultant of a family of n multilinear homogeneous polynomials of degree $k-1$, minus λ times a diagonal form of the same degree. By properties of the resultant, the degree of the characteristic polynomial is $n(k-1)^{n-1}$. Computing the resultant is known to be NP-hard over any field, in general ([11]). Thus, computing the characteristic polynomial of a hypergraph using traditional tools from commutative algebra is intractable. However, we can try to determine the characteristic polynomial of a hypergraph another way. Given the set of roots of a polynomial without multiplicity and an appropriate number of leading coefficients one can determine the multiplicity of its roots using the Faddeev-LeVerrier algorithm, a matrix form of the Newton Identities. **In [5] we provide a numerically stable algorithm for computing the multiplicity of the roots of a polynomial where the roots (without multiplicity) and some leading coefficients are known.** The algorithm is stable in the sense that if an eigenvalue is approximated by an ε-disk, where ε depends "reasonably" on the parameters of the problem, the resulting disk approximating its multiplicity contains exactly one integer. Our bound on ε is "reasonable" in that the number of bits required to approximate each root is proportional to the number of distinct roots of p and the logarithms of the ratio of the smallest difference of the roots with the largest difference of roots, the largest root, and the largest coefficient. The crux of the algorithm is a method to invert a Vandermonde matrix via a special factorization, when direct inversion would be numerically unstable.

We apply this algorithm to compute the adjacency characteristic polynomial of various hypergraphs. By Theorem 2 we can determine the leading coefficients of the characteristic polynomial, so it remains to determine the set of eigenvalues of the hypergraph. We can apply the aforementioned results to determine the spectrum (without multiplicities) of a hypertree. For a general hypergraph we can appeal to a method of Lu and Man to determine a subset of the set spectrum [13]. As an example, we compute the characteristic polynomial of the *Rowling* *hypergraph*, as shown in Figure 1, to be

$$\phi(\mathcal{R}) = x^{133}(x^3 - 1)^{27}(x^{15} - 13x^{12} + 65x^9 - 147x^6 + 157x^3 - 64)^{12} \cdot (x^6 - x^3 + 2)^6(x^6 - 17x^3 + 64)^3.$$

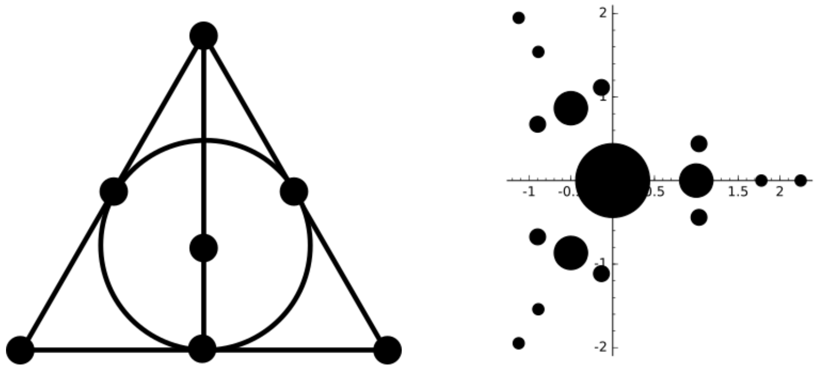

Figure 1: The Rowling hypergraph and its spectrum where a disk is centered at each root and its area is proportional to the root's multiplicity.

1.4 Future Work

Quasirandom Hypergraphs: I will apply Theorem 2 to prove a sufficient spectral condition for quasirandom hypergraphs. Intuitively, a hypergraph is *quasirandom* if it has the same number of copies of a particular subgraph as one would expect in a random graph (where, in its simplest form, each edge is taken with probability 1/2). This idea was first introduced for graphs in [3] and was later extended to hypergraphs in [2]. Generally speaking, the idea is that if one can show that a graph satisfies a particular condition then it is quasirandom. In [2] the authors show that a hypergraph is quasirandom if it has approximately the expected number of even partial octahedra (as described therein). Using Theorem 2 we aim to restate this condition in terms of the coefficients and perhaps the spectrum itself by showing that the linear combinations of subgraph counts appearing in the result are indeed "forcing sets" for quasirandomness.

Multiplicity of the Zero Eigenvalue: One can apply the Harary-Sachs Theorem to show that the multiplicity of the zero eigenvalue of a tree is equal to the size of its largest matching. I plan to prove a similar statement for hypertrees using the fact that the eigenvalues of a hypertree are the roots of a certain matching polynomial. I believe it is also possible to use Theorem 2 to show, by collecting summands corresponding to the same subgraph, that a hypergraph has a *coefficient threshold* which provides an upper-bound on the multiplicity of the zero eigenvalue. By determining the coefficient threshold of hypertrees and other classes of hypergraphs we could provide an upper-bound, or perhaps an explicit formula, for the multiplicity of the zero eigenvalue.

Open Source Software: Computing the adjacency characteristic polynomial of a hypergraph is NP-hard, in general. We have provided a numerically stable algorithm for computing the characteristic polynomial of a hypergraph given its set of eigenvalues and an tractable number of leading coefficients. I will continue working on each facet of this endeavor. This includes finding faster ways to compute the leading coefficients and store this information in an open access database, expanding on the Lu-Man method to provide an algorithm for determining eigenvalues of a larger class of hypergraphs, and utilizing high performance computing resources to perform these computations. In particular, I aim to compute the characteristic polynomial of the Fano Plane, especially since I know its first fifteen coefficients! I further plan on making this algorithm open source as a service to the mathematical community.

2 k-planar Crossing Numbers

A graph is *planar* if it can be drawn in the plane without edge crossings. Moreover, a graph is *k-planar* if its edges can be partitioned into k planes where the resulting subgraph in each plane is planar. This idea was first studied for biplanar drawings (i.e., $k = 2$) but gained attention when Tutte defined the *thickness* of a graph. The thickness of a graph G is the minimum number of planar graphs that G can be decomposed into [19]. This notion is relevant for VLSI chip design, where it corresponds to the number of layers required for realizing a network so that there are no wire crossings within a layer [15]. It has been shown that determining if a graph is biplanar is NP-complete in general [1, 14].

The traditional *crossing number* of a graph G, denoted by $cr(G)$, is the minimum number of edge crossings required to draw G in the plane. Owens defined the *biplanar crossing number* $cr_2(G)$ of G as the minimum sum of the crossing numbers of two graphs, G_0 and G_1, whose union is G [16]. The *k-planar crossing number of* G, denoted $cr_k(G)$, is similarly defined as the minimum sum of the crossing numbers of k graphs whose union is G. Trivially, $cr_k(G) \leq cr(G)$ as we can partition all the edges of G into one plane. Below I discuss my work collaborative work on improving known bounds for the k-planar crossing number.

2.1 The Biplanar Crossing Number of Hypercubes

Let Q_n denote the n dimensional hypercube whose vertices are binary strings of length n and where two vertices are incident if and only if they differ in precisely one bit. **It was shown in [8] that $cr_2(Q_8) \leq 256$, we have improved this bound to $cr_2(Q_8) \leq 128$ using a symmetric decomposition and further conjecture this bound to be exact [6].** Our approach highlights the relationship between symmetric drawings and the study of k-planar crossing numbers and can be extended naturally to higher dimensional hypercubes (i.e., $n > 8$). Our construction involves packing Q_8 with a particular subgraph which we will refer to as a *widget*. A drawing of a widget is given in Figure 2.1 where j and k are chosen from a particular set of binary words of length four. We prove our upper-bound by showing that this packing has the property that the edges of the widgets can be colored red and blue so that each vertex is incident to exactly one blue widget and one red widget. The edge bi-partition is given by partitioning the red edges into one plane and the blue edges into another. The desired bound follows from the observation that the crossing number of a widget is at most 8 (as shown in Figure 2.1) and that 16 widgets are needed to pack Q_8.

2.2 Bounding $cr_k(G)$ Given $cr(G)$

Czabarka, Sýkora, Székely, and Vrťo [9] proved that for every graph G we have

$$cr_2(G) \leq \frac{3}{8} cr(G).$$

Pach et al. [17] extended this investigation to the relationship between the k-planar crossing number and the (ordinary) crossing number of a graph. For every integer $k \geq 1$, they defined

$$\alpha_k = \sup \frac{cr_k(G)}{cr(G)},$$

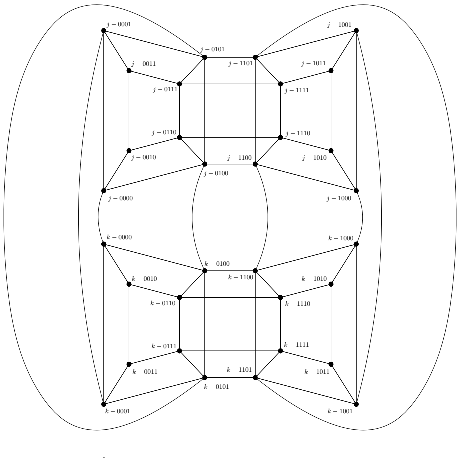

Figure 2: A widget where j and k are particular binary strings of length four.

where the supremum is taken over all *nonplanar* graphs G. They further proved that for every positive integer k,

$$\frac{1}{k^2} \leq \alpha_k \leq \frac{2}{k^2} - \frac{1}{k^3}.$$

Note that for $k = 2$, the two upper-bounds are identical. We have shown that the lower-bound of $1/k^2$ is asymptotically correct and have improved the upper-bound to the following.

Theorem 3. *$\alpha_k = \frac{1}{k^2}(1 + o(1))$ as $k \to \infty$. Moreover, $\frac{1}{k^2}$ is exact for the case of bipartite graphs.*

2.3 Future Work

Self-Complementary Drawings: Our biplanar drawing of Q_8 has the property that the graphs in each plane are isomorphic. This is a rather special property and is termed *self-complementary* in [8]. It is unclear if requiring a drawing to be self-complementary can create more crossings. In particular, such symmetry would be expected when considering a highly symmetric graph like the hypercube. I will continue investigating this question by determining if our bound on $cr_2(Q_8)$ is sharp.

Improved Bounds on α_k: Theorem 3 and its proof surrender control over the $o(1)$ term. We have determined an improved upper-bound for the case of $3 \leq k \leq 10$. It would be interesting if a stronger statement could be made by restricting the supremum to be over a particular family of graphs. This relates to the previous question if we consider only self-complementary drawings.

3 Modeling Dark Net Markets

Digital black markets operating in the dark web have been explosive in their growth and proliferation, with 16 concurrent major markets currently in operation with 171 major markets opening and closing since law enforcement shuttered the original Silk Road cryptomarket in 2013 [22]. Combating this phenomenon is costly, with the UK pledging £9 million to fight dark web markets in 2018 [21]. However, evidence suggests that these markets are adaptive to the machinations of law enforcement. In particular, large-scale operations designed to disrupt marketplaces are ineffective despite observable marketplace closures [10]. As such, understanding the mechanisms by which the broader black market persists is crucial to the development of effective countermeasures.

Dark net markets operate under an escrow service wherein a broker oversees all transactions. We have proven that, unlike any known market or business (including legal markets which operate under escrow), these cryptomarkets are designed to self-terminate and go dark. **We have proven that a dark net market will opt to self-terminate and have given a formula to predict the time of closure given the broker's transaction fee, the expected growth of the market, and the risk-free rate.** This work shows that every dark net market is a scam in the sense that its profit maximization is derived from the broker closing the market and stealing the funds at an optimal point in its operation. Our formulaic approach gives expectations of the lifespan on these marketplaces which agree with empirical observation. We additionally show that law enforcement activity serves to accelerate this closure process by changing the level of risk in the environment.

3.1 Future Work

Identifying Users in a Dark Net Market: Digital black markets organize differently than their legal counterparts. Roughly speaking, their social network forms a star where the central user (the broker) is compensated for assuming the risk of other users. I am eager to apply my research in spectral hypergraph theory to provide a richer model for digital black markets which differentiates users based on behavior. It is my hope that this research would better enable law enforcement to allocate their resources to accelerate market closure.

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